The Gaussian Interference Channel at Low SNR

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Abstract — **For the two-user Gaussian interference channel we show that, up to second order in the signal-to-noise ratio (SNR), the mutual information between each transmitter and intended receiver depends only on the covariance matrices of the input distributions. Our analysis suggests that in the low SNR regime there is an interference threshold above which it is better for the users to alternate signal transmission than to transmit simultaneously.**

I. Introduction and Model

We are interested in evaluating, up to second order in the SNR, the sum-rate capacity of the two-user Gaussian interference channel. Over n channel uses, the received signals Y_1 and Y_2 are related to the transmitted signals X_1 and X_2 (all n-dimensional vectors) by the equations

$$\begin{aligned} Y_1 &= \sqrt{\rho}(h_{11}X_1 + h_{21}X_2) + V_1 \\ Y_2 &= \sqrt{\rho}(h_{12}X_1 + h_{22}X_2) + V_2. \end{aligned} \tag{1}$$

Here the propagation parameters h_{11}, h_{21}, h_{12} and h_{22} are assumed to be real, fixed during transmission and known to both transmitter and receiver. The transmit signals X_1 and X_2 are assumed to have zero mean in each entry and have the power constraint $\mathbf{E}\mathrm{tr}X_1X_1^T = \mathbf{E}\mathrm{tr}X_2X_2^T \leq n$. The noise variables V_1 and V_2 are each independent real-valued Gaussian random variables with zero mean, unit variance and independent entries. This makes the normalization parameter ρ directly proportional to the SNR at each receiver.

Ahlswede in [1] gives a limiting expression for the set of achievable rates (R_1, R_2) for this channel. From this, the sum-rate capacity is given by

$$A(\rho) = \lim_{n\to\infty} \sup_{P_{X_1X_2}=P_{X_1}P_{X_2}} S(\rho), \tag{2}$$

where $S(\rho) := \frac{1}{n}\left(I(X_1;Y_1) + I(X_2;Y_2)\right)$. We are interested in the second order expansion $A(\rho) = a_1\rho + a_2\rho^2 + o(\rho^2)$ where

$$\begin{aligned} a_1 &= \lim_{\rho\to 0} \frac{\lim_{n\to\infty}\sup_{P_{X_1X_2}=P_{X_1}P_{X_2}} S(\rho)}{\rho}, \\ a_2 &= \lim_{\rho\to 0} \frac{\lim_{n\to\infty}\sup_{P_{X_1X_2}=P_{X_1}P_{X_2}} S(\rho) - a_1\rho}{\rho^2}. \end{aligned} \tag{3}$$

Thus, computing $A(\rho)$ to second order requires us to compute the supremum over the probability distributions (and let $n \to \infty$) before letting $\rho \to 0$. This makes the problem as difficult as computing $A(\rho)$ itself. To simplify the problem, we shall perform the limit $\rho \to 0$ first and compute:

$$\begin{aligned} b_1 &= \lim_{n\to\infty} \sup_{P_{X_1X_2}=P_{X_1}P_{X_2}} \lim_{\rho\to 0} \frac{S(\rho)}{\rho}, \\ b_2 &= \lim_{n\to\infty} \sup_{P_{X_1X_2}=P_{X_1}P_{X_2}} \lim_{\rho\to 0} \frac{S(\rho) - b_1\rho}{\rho^2}, \end{aligned}$$

Now the supremum becomes one over the covariance matrices of X_1 and X_2 where we have used b_1 and b_2, rather than a_1 and a_2, cognizant of the fact that the interchange of the order of the limits may not be justified.

II. Main Results

Theorem 1. *Consider the Gaussian interference channel model (1) and assume that the distributions of X_1 and X_2 are even. Then to second order*

$$I(X_1;Y_1) \approx \frac{\rho}{2}h_{11}^2\mathrm{tr}R_{X_1} - \frac{\rho^2}{4}\left(h_{11}^4\mathrm{tr}R_{X_1}^2 + 2h_{11}^2h_{21}^2\mathrm{tr}R_{X_1}R_{X_2}\right),$$

where $R_{X_1} := \mathbf{E}X_1X_1^T$ and $R_{X_2} := \mathbf{E}X_2X_2^T$ are the $n \times n$ covariance matrices of X_1 and X_2 respectively.

Whereas one may have suspected that the mutual information would have depended on fourth order statistics, the above shows that it depends only on the input covariance matrices. We may then write $I(X_1;Y_1) + I(X_2;Y_2)$ up to second order in ρ and then optimize this over choices of R_{X_1} and R_{X_2} satisfying the power constraint. This leads to the following result.

Theorem 2. *For the two-user Gaussian interference channel described by (1) with the given power constraint let $\eta := (h_{21}/h_{22})^2 + (h_{12}/h_{11})^2$. Then to maximize $I(X_1;Y_1) + I(X_2;Y_2)$ up to second order in the SNR,*

- *if $\eta \leq 1$ (low interference), both users should each transmit simultaneously signals of unit variance.*
- *if $\eta > 1$ (high interference), the users should alternate transmission with user i transmitting for a fraction $h_{ii}^2/(h_{11}^2 + h_{22}^2)$ of the time, signals with variance $(h_{11}^2 + h_{22}^2)/h_{ii}^2$.*

In [2] Costa defines four interference levels (weak, moderate, strong, very strong) for which expressions for the maximum sum-rate are given. The strong and very strong interference results are known to be exact while the weak and moderate interference cases are the best known sum rates. Our analysis corresponds with the weak interference case when $\eta \leq 1$ and the moderate interference case when $\eta > 1$, both to second order in ρ. However the strong and very strong interference regimes (where interference cancellation is the decoding strategy) lead to a higher sum-rate than is achieved in our analysis. In this high interference regime then it appears that expanding mutual information to second order before the optimization is invalid. Whether the low-SNR sum-rate can be found without resorting to a direct evaluation of (3) remains a question of interest.

References

[1] R. Ahlswede. Multi-way communication channels. *Proc. 1971 IEEE Int. Symp. Information Theory*, pages 23–52.

[2] M. Costa. On the Gaussian Interference Channel *IEEE Trans. Inform. Theory*, 31:607–615, Sept. 1985.

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